Exercise 58

(a) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.

(b) If $a \neq 0$, find $g'(a)$.

(c) Show that $y = x^{2/3}$ has a vertical tangent line at $(0, 0)$.

(d) Illustrate part (c) by graphing $y = x^{2/3}$.

Solution

Use Equation 2.7.5 to find $g'(a)$, assuming $a \neq 0$.

$$\begin{aligned}
g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} &= \lim_{x \to a} \frac{x^{2/3} - a^{2/3}}{x - a} \\
&= \lim_{x \to a} \frac{\left(x^{1/3}\right)^2 - \left(a^{1/3}\right)^2}{x - a} \\
&= \lim_{x \to a} \frac{\left(x^{1/3} + a^{1/3}\right)\left(x^{1/3} - a^{1/3}\right)}{x - a} \\
&= \lim_{x \to a} \frac{\left(x^{1/3} + a^{1/3}\right)\left(x^{1/3} - a^{1/3}\right)}{\left(x^{1/3} - a^{1/3}\right)\left(x^{2/3} + a^{1/3}x^{1/3} + a^{2/3}\right)} \\
&= \lim_{x \to a} \frac{x^{1/3} + a^{1/3}}{x^{2/3} + a^{1/3}x^{1/3} + a^{2/3}} \\
&= \frac{a^{1/3} + a^{1/3}}{a^{2/3} + a^{1/3}a^{1/3} + a^{2/3}} \\
&= \frac{2a^{1/3}}{3a^{2/3}} \\
&= \frac{2}{3}a^{-1/3}
\end{aligned}$$

$g'(0)$ does not exist because

$$\begin{aligned}
g'(0) &= \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} \\
&= \lim_{x \to 0} \frac{x^{2/3} - 0^{2/3}}{x} \\
&= \lim_{x \to 0} \frac{x^{2/3}}{x} \\
&= \lim_{x \to 0} \frac{1}{x^{1/3}} \\
&= \pm\infty
\end{aligned}$$

this limit does not exist. Note that $g'(0)$ represents the slope of $g(x)$ at $x = 0$. Since $g'(0) = \pm\infty$, there is a vertical tangent at $x = 0$.

Below is a graph of $g(x) = x^{2/3}$ to illustrate the vertical tangent at $x = 0$.